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A Variational Principle for Periodic Waves of Infinite Depth

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A VARIATIONAL PRINCIPLE FOR PERIODIC WAVES OF INFINITE DEPTH

Ellen R. Gottlieb

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CONTENTS

ABSTRACT:

This paper deals with periodic waves on an ocean of infinite depth. The flow is assumed to be two-dimensional, incompressible, steady, and irrotational. The impossibility of the existence of an asymmetric wave is proved. This is accomplished through an application of Steiner symmetrization. Also discussed is the shape of possible periodic waves.

Using the calculus of variations, we set up an extremal problem involving the kinetic energy, an area integral, and the potential energy. For waves of small amplitude the kinetic energy is shown to be a minimum if we fix the area and the potential energy. This is accomplished by showing the first variation to be zero and the second variation to be positive. Since the kinetic energy is closely related to the Dirichlet integral, this is a generalization of the Dirichlet principle. This result is applicable in showing the existence of periodic surface waves.

CHAPTER 1

INTRODUCTION

In this paper we shall discuss periodic waves at the free surface of an ocean of infinite depth. The motion of the fluid is assumed to be two-dimensional, incompressible, and irrotational. The only external force acting on the fluid is the constant force of gravity.

We shall use a moving coordinate system which travels with the same velocity as the wave. With respect to this coordinate system, the flow that we are investigating can be considered to be steady.

Since the motion is irrotational, a velocity potential $\phi(x,y)$ exists. If u, v are the components of velocity parallel to the x and y coordinate axes respectively at the point (x,y), then $\phi_x = u$ and $\phi_y = v$. The equation of continuity (as expressed for incompressible fluids),

$$\operatorname{div}(u,v) = 0 ,$$

then tells us that ϕ is harmonic.

Since ϕ is harmonic, we know ϕ has a harmonic conjugate, which we call ψ. The curves ψ = constant are the streamlines and we may let $\psi = 0$ represent the free surface. The function $\zeta(z) = \phi(x,y) + i\psi(x,y)$ is called the complex potential. The

speed of the flow, which we denote by q, is then defined by the equation

$$q = \left|\frac{d\zeta}{dz}\right| .$$

The shape of the free surface is determined by the free boundary condition

$$q^2 + 2gy = \lambda ,$$

where g is the acceleration of gravity and λ is a constant. This condition is derived from Bernoulli's law

$$\frac{q^2}{2} + gy + \frac{p}{\rho} = \text{const.} ,$$

where p is the pressure and ρ is the density at the point (x,y). Our free boundary condition follows by noting that along this curve the pressure is constant.

To facilitate our investigations we will assume that the flow has period one with respect to the x-direction and confine ourselves to one period D. The semi-infinite region D is assumed to have vertical side boundaries and an upper boundary which we will denote by C.

Periodic surface waves on an ocean of infinite depth have been investigated by many mathematicians, including Nekrassov, Levi-Civita and Lichtenstein. The existence of such waves was first proved in 1921 by Nekrassov [13]. In 1925, Levi-Civita [1] published an existence and uniqueness proof for the case

in which the quantity $\frac{g\omega}{2\pi c^2}$ is sufficiently close to unity, where ω is the wave length and c is the speed of the wave. Levi-Civita proved the convergence of a power series in the amplitude of the solution. The first term of this series is the linearized solution (see Lamb for a discussion of the linear theory [9]). In 1931, Lichtenstein used the theory of non-linear integral equations to demonstrate existence and uniqueness of a solution if

$$\frac{g\omega}{2\pi c^2} = m-k \ ,$$

where m is a positive integer and k is sufficiently small.

More recent results are described in the appendix to Water Waves by Stoker [16]. Martin has also contributed to this subject, and in particular in a paper [3] written with Dunninger in 1966 he discusses the question of uniqueness.

In this paper we shall investigate periodic surface waves on an ocean of infinite depth by employing a variational principle. The first sections deal with the construction of suitable functionals which we will need. Specifically, we will introduce the kinetic energy associated with the disturbed part of the flow, which is given by

$$M = \iint_D [\nabla(\psi - cy)]^2 \, dA \ ,$$

where c is the speed at $y = -\infty$. We also consider certain finite area and potential energy integrals

$$A = \iint dA$$

and

$$E = 2g \iint y \, dA \ .$$

We will prove the following variational formula:

$$\delta(M + (\lambda - c^2)A - E) = \int_C (q^2 + 2gy - \lambda)\delta n ds \ ,$$

which shows that the free boundary condition we obtained from Bernoulli's law is a transversality condition for the extremal problem

$$M + (\lambda - c^2)A - E = \text{minimum} \ .$$

More precisely, from the above formula we see that if A and E are held fixed, then M has a stationary value when C describes a wave.

The functional $M + (\lambda - c^2)A - E$ is used to prove the main result of the paper, which is the derivation of a uniqueness theorem excluding the existence of certain asymmetric waves. We will apply continuous Steiner symmetrization (cf. [14]) with respect to the y-axis to the above functional. From this it will be shown that the functional is convex with respect to continuous Steiner symmetrization. We then conclude that the wave must be symmetric with respect to a vertical line through a trough or crest.

We will formulate a more specific problem in the calculus of variations for ψ. In the case of Levi-Civita's waves of sufficiently small finite amplitude, it will be shown that the wave minimizes M for fixed A and E. However, we have not as yet been able to prove this result for the general case.

Finally we prove that the free boundary of the wave has a non-parametric representation $y = y(x)$, where $y(x)$ is a single-valued function of x with no vertical tangent at any point. To prove this theorem, we make use of Hopf's lemma, which gives a strict inequality for the normal derivative of a subharmonic function when evaluated at a boundary point at which a maximum occurs.

In 1965 Garabedian [5] published a paper on periodic surface waves on an ocean of finite depth in which he formulates an extremal problem in order to prove the existence of such a wave. Since the wave described by ψ for this case is a minimax solution rather than a minimum, the use of Morse theory was necessary.

The results of this paper may be used as a basis for an existence theorem in the case of an ocean of infinite depth. In particular, the result that the wave minimizes M for fixed A and E shows that Morse theory can be avoided. Our theorem that the wave has a non-parametric representation $y = y(x)$ can be used in restricting the class of competing functions involved in solving the minimum problem to those which have this same representation. Symmetrization yields a corresponding result

[illegible]. It tells us that if we restrict our-selves to one period of the wave, then the free surface has a non-parametric representation $x = x(y)$ with just two branches. Hence we may restrict the class of competing functions to those which can be described in the same way. Such an existence proof by variational methods, however, is beyond the scope of this paper.

CHAPTER 2

ESTABLISHING FUNCTIONALS FOR THE INVESTIGATION OF SURFACE WAVES

In this chapter we look at the behavior of the stream function ψ as $y \to -\infty$ and then set up suitable functionals with which to investigate periodic surface waves on an ocean of infinite depth.

In the case of an ocean of finite depth, Garabedian [5] used the Dirichlet integral as a functional for his investigations of the existence and uniqueness of surface waves. Since the Dirichlet integral for the case of an infinitely deep ocean may diverge, we must alter it to obtain a convergent integral suitable for our work. To do this we must first discuss the behavior of certain harmonic functions ψ defined on a semi-infinite region D, whose upper boundary is a curve C and whose side boundaries are the two vertical lines, $x = 0$ and $x = 1$.

2.I. Asymptotic Behavior at Great Depth

Theorem 2.I. Let ψ be defined on a semi-infinite domain D whose upper boundary is the curve C and which is bounded on the sides by the vertical lines $x = 0$ and $x = 1$ (see Fig. 1). We assume ψ has the following properties:

$$\psi = 0$$

on C,

$$\psi(0,y) = \psi(1,y)$$

for all ψ.

$$\Delta\psi = 0$$

in D and on the boundary of D, and

$$(\nabla\psi)^2 = O(1)$$

as $y \to -\infty$.

Through the techniques of conformal mapping, we show that ψ_x and ψ_y approach constant values exponentially fast as $y \to -\infty$.

Proof: We may map the strip $0 < x < 1$ onto the w-plane ($w = u + iv$), slit along the line $v = 0$, making $-\infty$ in the z-plane correspond to zero in the w-plane. The curve C in the z-plane (see Fig. 1) maps onto a finite curve C' in the w-plane (see Fig. 2). This map is effected by

$$w = e^{-2\pi iz} .$$

Note that the line $x = 1$ is mapped into the upper part of the cut and $x = 0$ is mapped into the lower part of the cut.

Let $\zeta(z) = \phi + i\psi$, where ϕ is the harmonic conjugate of ψ. Then $\zeta(z)$ is analytic in D and on the boundary of D. The function

$$\zeta'(z) = \frac{d\zeta}{dz}$$

may be written as

$$\zeta'(z) = \phi_x + i\psi_x$$

or

$$\zeta'(z) = \psi_y + i\psi_x .$$

We note that $\zeta'(z)$ is also analytic in D.

Now we can solve for z as an analytic function of w in the slit w-plane. Hence

$$g(w) = \zeta'(z(w)) = \zeta'(z)$$

is an analytic function of w in a bounded region D' of the w-plane. This region may be visualized as the shaded region of Fig. 2.

Let us look at what happens as $v \to 0$ through positive and negative values of v. Since ψ is periodic, we have

$$\psi_x(0,y) = \psi_x(1,y) \ .$$

Thus

$$\lim_{v\to 0^+} g(w) = \lim_{v\to 0^-} g(w)$$

for $u \neq 0$. This implies that g(w) may be extended to the points on the slit bordering the domain D' in the w-plane, except possibly the origin. We assumed

$$|g(w)|^2 = |\zeta'(z)| \leq K^2$$

near $w = 0$ or $y = -\infty$. Hence g(w) has at most a removable singularity at $w = 0$. To make g(w) analytic at $w = 0$, we put

$$a_o = g(0) = \lim_{w\to 0} g(w) \ .$$

Then, it follows from Taylor's theorem that

$$g(w) = \sum_{n=0}^{\infty} a_n w^n$$

in a neighborhood of $w = 0$. Therefore,

$$\zeta'(z) = g(w) = \sum_{n=0}^{\infty} a_n e^{-2\pi i n z}$$

for y sufficiently large and negative. Since $\zeta'(z) = \psi_y + i\psi_x$, we arrive at the expansions

$$\psi_y = \mathrm{Re}\ a_o + \mathrm{Re} \sum_{n=1}^{\infty} a_n e^{-2\pi i n z}$$

and

$$\psi_x = \mathrm{Im}\ a_o + \mathrm{Im} \sum_{n=1}^{\infty} a_n e^{-2\pi i n z} \ .$$

Thus we arrive at the desired result, i.e.,

$$\psi_y = \mathrm{Re}(a_o) + O(e^{2\pi y})$$

and

$$\psi_x = \mathrm{Im}(a_o) + O(e^{2\pi y})$$

as $y \to -\infty$. In other words, the horizontal and vertical components of velocity approach constants exponentially fast.

2.II.a. Introduction of the Quantity M

Let ψ be a stream function defined on the domain D, which is the same domain as in previous sections (see Fig. 1). Let us assume the following for ψ:

$$\psi(0,y) = \psi(1,y)$$

for all y,

$$|\nabla\psi|^2 = O(1)$$

as $y \to -\infty$, and

$$\psi = 0 \text{ on the free surface} .$$

Then it follows from Theorem 2.I, that ψ has the following expansion near $y = -\infty$:

$$\psi = cy + y_o + \sum_{n=1}^{\infty} a_n e^{2\pi n y} \cos(2\pi n x + \phi_n) .$$

Note that the a_n here are not quite the same as the a_n used previously.

Since we have investigated the behavior of ψ near $y = -\infty$, we can now introduce the quantity M. This quantity M will represent the kinetic energy associated with the disturbed part of the flow and will be an integral involving the term $(\nabla\psi)^2$ plus other terms which are added on to make it convergent over the domain D. More precisely, we define

$$M = \iint_D [\nabla(\psi - cy)]^2 \, dA .$$

It is easily seen that this integral is convergent, since we have investigated the behavior of ψ and $(\nabla\psi)^2$ near $y = -\infty$. After subtracting the term cy from ψ, we see that the square of the gradient of the resulting function dies out exponentially as $y \to -\infty$.

Most of the results in this paper rest upon the investigation of properties of M. Formulas for the first and second variations of M will be derived. The functional M and its first variation δM will be important in showing the impossibility of the existence of certain asymmetric periodic waves. In the special case in which the wave is almost flat, we indicate in Chapter 4 how M might be used to set up an existence proof for periodic waves in an ocean of infinite depth. In particular we show that for a fixed area and a certain fixed potential energy, the second variation of M is positive.

2.II.1. The First Variation of M

Suppose

$$J(\varepsilon) = \iint_D F(x,y,\psi,\psi_x,\psi_y)\,dA \; ,$$

where the domain D and the functions ψ, ψ_x, and ψ_y depend on the parameter ε, and ε is small. Then the first variation of the functional $J(\varepsilon)$ can be defined by the formula

$$\delta J = \left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon \; .$$

Let D be the same semi-infinite domain as mentioned in previous sections. Its upper boundary is C; its side boundaries are the vertical lines $x = 0$ and $x = 1$. We obtain the domain D^* from D by varying only the upper boundary of D. The upper

boundary C^* of D^* is obtained by making a variation of the form

$$\delta n = \varepsilon\rho(s)$$

along the inner normal to C with respect to the domain D. The function $\rho(s)$ is assumed to be a real analytic function of the arc length parameter s on C.

It is assumed that ψ is a symmetric stream function, that is,

$$\Delta\psi = 0$$

in D,

$$\psi = 0$$

on C,

$$\psi(0,y) = \psi(1,y)$$

for all y, and

$$(\nabla\psi)^2 = O(1)$$

as $y \to -\infty$.

Let ψ^* be the harmonic function defined on D^* which satisfies the analogous conditions with respect to D^*, that is,

$$\Delta\psi^* = 0$$

in D^*,

$$\psi^* = 0$$

on C^*,

$$\psi^*(0,y) = \psi^*(1,y)$$

for all y, and

$$([illegible]) \qquad (1)$$

as $y \to -\infty$.

We note that the curve C is assumed to be analytic and that the variation is analytic. This implies that C^* is analytic. Hence if C and C^* are sufficiently close, then by Schwarz's principle of reflection ψ and ψ^* can both be extended outside of their domains, if necessary, so that they are both harmonic in D and D^* and on C and C^*.

Suppose, at first, we make the assumption that D^* lies entirely in the interior of D. Let $\vec{n}$ be the inner normal to D, and let D_k be

$$D \cap \{y \mid y \geq -k,\ k > 0\} .$$

Then we may rewrite M as

$$M = \lim_{k \to \infty} \iint_{D_k} [\nabla(\psi - cy)]^2 \, dA .$$

We have through an application of Green's theorem

$$M = \int_C cy \frac{\partial}{\partial n} (\psi - cy) ds ,$$

since the line integrals over the vertical boundaries of D_k cancel each other and since, in the limit as $k \to \infty$, the line integral over $y = -k$ vanishes.

Let A be the area bounded by the curve C and the x-axis. Then M can be written as

$$M = c\int_C y\,\frac{\partial\psi}{\partial n}\,ds + c^2A\ ,$$

and therefore

$$M = c\left[\int_C \left(y\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial y}{\partial n}\right)ds\right] + c^2A\ .$$

This follows because $\psi = 0$ on the curve C and hence the second integral on the right of the above equation is zero. Now

$$\iint_{D-D^*} \nabla\psi\cdot\nabla\psi^* dA = \int_C (\nabla\psi)^2\ \delta n ds + O(\varepsilon^2)\ .$$

On the other hand, by Green's theorem,

$$\iint_{D-D^*} \nabla\psi\cdot\nabla\psi^* dA = -\int_C \psi^*\,\frac{\partial\psi}{\partial n}\,ds + \int_{C^*} \psi^*\,\frac{\partial\psi}{\partial n}\,ds\ .$$

The second integral on the right side of the equation vanishes on C^*. Adding $\int_C \psi\,\frac{\partial\psi^*}{\partial n}\,ds$ to the first integral on the right hand side of the equation, which leaves it unchanged because $\psi = 0$ on C, we get

$$\iint_{D-D^*} \nabla\psi\cdot\nabla\psi^* dA = -\int_C \left(\psi^*\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\psi^*}{\partial n}\right)ds$$

$$= \int_{y=-k} \left(\psi^*\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\psi^*}{\partial n}\right)ds\ .$$

Now

$$\psi^* = y_o^* + cy + \sum_{n=1}^{\infty} a_n^* e^{n\pi y} \cos\,(n\pi x + \phi_n^*)$$

and

$$\psi = y_o + cy + \sum_{n=1}^{\infty} a_n e^{n\pi y} \cos\,(n\pi x + \phi_n)$$

[illegible] $\frac{\partial \psi}{\partial n}$, [illegible] and $\frac{\partial \psi^{\bullet}}{\partial n}$, and [illegible] higher order terms of the expansion, we have

$$\iint_{[illegible]} \nabla\psi \cdot \nabla\psi^{\bullet}\, dA = \int_{y=-k}^{1} \{-[y_0^{\bullet} + cy]c + [y_0 + cy]c\}\, dx$$

$$= -c(y_0^{\bullet} - y_0) \; .$$

We now derive the formula

$$M - c^2 A = -c y_0 \; ,$$

which relates the kinetic energy M with the coefficient y_0 in the Taylor expansion for the stream function ψ near $y = -\infty$. An application of this formula allows us to get the variational formula.

We have

$$M = c\left[\int \psi \frac{\partial y}{\partial n}\, ds + \int -\psi \frac{\partial y}{\partial n}\, ds\right] + c^2 A \; .$$

Then, using Green's theorem and taking into account that y and ψ are harmonic in Γ, we obtain

$$M = c\left[\int_{y=-k}^{1} \left(-y \frac{\partial \psi}{\partial n} + \psi \frac{\partial y}{\partial n}\right) dx\right] + c^2 A \; .$$

Substituting the above expansion for ψ near $y = -\infty$, i.e., for k sufficiently large, gives

$$M = c\left[\int_{\substack{0\\ y=-k}}^{1} \{(-y)(-c) - (cy + y_o)\}dx\right] + c^2A ,$$

which reduces to

$$M = -cy_o + c^2A ,$$

or

$$M - c^2A = -cy_o .$$

Since we may write the analogous formula

$$M^* - c^2A^* = -cy_o^* ,$$

the total variation of $M - c^2A$ is

$$(M^* - M) - c^2(A^* - A) = -c(y_o^* - y_o) .$$

Also, we have

$$M^* - M = -c(y^* - y) + c^2(A^* - A) .$$

From the equations

$$\iint_{D-D^*} \nabla\psi\cdot\nabla\psi^* dA = \int_C (\nabla\psi)^2 \delta n ds + O(\varepsilon^2)$$

and

$$\iint_{D-D^*} \nabla\psi\cdot\nabla\psi^* dA = -c(y_o^* - y_o) ,$$

we have

$$-c(y_o^* - y_o) = \int_C (\nabla\psi)^2 \delta n ds + O(\varepsilon^2) .$$

Therefore

$$M^* - M = c^2(A^* - A) \; [illegible]$$

$$\int_C q^2(\delta n)ds + O(\epsilon^2) \; .$$

where q is the speed along C. Denoting the first variation by the symbol δ, and noting that

$$\delta A = - \int_C (\delta n)ds \; ,$$

we obtain the final result

$$\delta M = \int_C (q^2 - c^2)\delta n ds \; .$$

We have assumed that the domain D^* is interior to D. However, this is not necessary. If one of the domains is not inside the other, we consider a third domain D^{**} which has the same vertical side boundaries as D and D^* but is contained within both D and D^*. Let C^{**} denote the upper boundary of D^{**}. We have, by the above computation, expressions for $M^{**} - M^*$ and $M^{**} - M$ up to first order. If we denote the variation along the inner normal to C^* by the symbol $\tilde{\delta} n$ and the variation along the inner normal to C by the symbol $\tilde{\tilde{\delta}} n$, where $\tilde{\delta} n$ and $\tilde{\tilde{\delta}} n$ both bring us to a point on the curve C^{**}, then we can write δn as

$$\delta n = \tilde{\tilde{\delta}} n - \tilde{\delta} n + O(\epsilon^2) \; .$$

To first order in [illegible], we have

$$M^{**} - M = \int_C (q^2 - c^2)\tilde{\tilde{\delta}} n ds$$

and

$$M^{**} - M^* = \int_{C^*} (q^{*2} - c^2)\tilde{\delta} n ds \ .$$

We can rewrite the second equation in the form

$$M^{**} - M^* = \int_C (q^2 - c^2)\tilde{\delta} n ds + O(\varepsilon^2) \ .$$

Then

$$M^* - M = (M^{**} - M) - (M^{**} - M^*)$$

$$= \int_C (q^2 - c^2)\delta n ds + O(\varepsilon^2) \ .$$

Hence the first variation of M is given by

Theorem 2.II.

$$\delta M = \int_C (q^2 - c^2)\delta n ds \ .$$

2.II.c. The First Variation of M for a Wave

In Section 2.II.b., we derived the formula

$$\delta M = \int_C (q^2 - c^2)\delta n ds$$

for the first variation. To derive this formula, we made use of the functional

$$A = \iint \chi(x,y) dA \ ,$$

where

$$\chi(x,y) = 1$$

for $y > 0$ and

$$\chi(x,y) = 0$$

for $y \leq 0$, and the first variation of A

$$\delta A = -\int_C \delta n ds .$$

We now assume that ψ describes a wave and define the functional

$$F(s) = M + (\lambda - c^2)A - E ,$$

where

$$E = \iint 2gy\ \chi(x,y) dA .$$

The integrals M, A, and E are defined over the domain D^*, which depends upon s. This domain is described in detail in Section [illegible]. The integral E represents the potential energy of the flow. The first variation of E is obviously given by

$$\delta E = -\int_C 2gy\ \delta n ds .$$

Since C is the free boundary of the wave, we no longer have to require that C be analytic in order to derive the variational formulas, because the requirement can be deduced from the free boundary condition [11]

$$q^2 + 2gy = \lambda .$$

Since we know δA, δM, and δE, we may now write down an expression for the first variation of F. This is given by the formula

$$\delta F = \int_C (q^2 + 2gy - \lambda)\delta n ds \ .$$

Therefore a necessary condition for $F(\varepsilon)$ to have a stationary point when $\varepsilon = 0$ is that

$$q^2 + 2gy = \lambda \ ,$$

which is the free boundary condition. In other words ψ satisfies the free boundary condition and hence, describes a wave. This result is the basis for the uniqueness theorem of the next chapter and for the setting up of the variational principle of the concluding chapter.

CHAPTER 3

THE SHAPE OF PERIODIC WAVES

This chapter deals with the possible shapes of periodic waves. If we assume that all the level curves $\psi(x,y)$ = constant rise and fall just once in one period, it will follow, from an application of Steiner symmetrization, that no asymmetric periodic wave is possible. This is the main theorem of this section.

By assuming that the free surface of the wave $\psi = 0$ rises and falls just once in one period we can show that the curves $\psi(x,y) = t$ have the same property providing t is sufficiently large and negative. Thus the hypothesis of the theorem may be weakened. With an additional restriction on the free surface slope we may also prove that all curves $\psi = t$ rise and fall precisely once in one period.

It will also be shown that a periodic surface wave can be described non-parametrically by a single-valued function $y = y(x)$ which has no vertical tangent. This last result will depend on Hopf's lemma for subharmonic functions, which deals with the normal derivative at a point on the boundary of a domain at which a maximum occurs [1]. These results have an application to the study of existence and uniqueness of surface waves by variational methods.

3.I. The Non-Existence of Certain Periodic Asymmetric Waves

We proceed to prove the non-existence of periodic asymmetric waves under the assumption that all the streamlines ascend and descend just once in a single period. In other words, we assume that each streamline $\psi(x,y)$ = constant has a non-parametric representation $x = x_j(y)$ in the domain D describing one period of the flow, with just two branches x_1 and x_2.

Motivated by Section 2.II.c., we shall set up a functional F which will depend on a parameter θ, where $0 \leq \theta \leq 2$. In this interval $F(\theta)$ will be shown to be convex and the derivative of $F(\theta)$ will be shown to vanish for $\theta = 0$ and $\theta = 2$. Thus it will be proven that $F(\theta)$ must be constant. This conclusion will result in a contradiction unless the streamlines ψ = constant are symmetric with respect to a vertical line through a crest.

We recall that D is the semi-infinite domain which lies between $x = 0$ and $x = 1$ as in previous sections. In this section, the function ψ defined on D is assumed to describe a wave. For now, we translate the y-axis to the right by 1/2 (see Fig. 3).

Suppose $\psi(x,y)$ is asymmetric with respect to the new y-axis. Then $\psi(x,y)$ and $\psi(-x,y)$ define different streamlines. This means that

$$x_1'(y) \neq -x_2'(y) \ .$$

We proceed by constructing a new set of streamlines with two branches $x = X_1(y)$ and $x = X_2(y)$ defined by

$$
(1) \qquad \begin{cases} X_1 = x_1 - \theta\left(\dfrac{x_1 + x_2}{2}\right) \\[2ex] X_2 = x_2 - \theta\left(\dfrac{x_1 + x_2}{2}\right) \end{cases} .
$$

where $0 \leq \theta \leq 2$.

For each value of θ, we may define a function $\psi(\theta)$ whose level curves have the branches X_1 and X_2 corresponding to the same value of θ. The function $\psi(\theta)$ is defined on the domain D_θ with the same vertical side boundaries as D and with the curve $\psi(\theta) = 0$ for $-1/2 \leq x \leq 1/2$ as upper boundary. Note that $\psi(\theta)$ is not necessarily harmonic. We may interpret the transformation (1) as continuous Steiner symmetrization. For $\theta = 0$, we have the identity transformation. For $\theta = 1$, the transformation is Steiner symmetrization with respect to the y-axis. For $\theta = 2$, $\psi(x,y)$ is mapped into $\psi(-x,y)$.

Let

$$
F(\theta) = M + (1-c^2)A - E
$$

$$
= \iint_{D_\theta} \left[[\nabla(\psi(\theta) - cy)]^2 + (1-c^2)\chi(x,y) - 2gy\,\chi(x,y) \right] dA \; .
$$

where, again,

$$
\chi(x,y) = 1
$$

for $y > 0$, and

$$
\chi(x,y) = 0
$$

for $y \leq 0$. This functional $F(\theta)$ is well-defined provided we show that $M(\theta)$ converges for each θ, where $0 \leq \theta \leq 2$. The

convergence of $M(\theta)$ will be shown trivially once we have shown it to be convex.

The crucial part of this proof consists in showing that $F(\theta)$ is convex. It will be convenient to use the following formulation of convexity: $F(\theta)$ is convex if

$$F\left(\frac{\theta_1 + \theta_2}{2}\right) \leq \frac{1}{2} [F(\theta_1) + F(\theta_2)] .$$

The functionals A and E are obviously independent of θ, because the transformation (1) is simply a horizontal shift of the curves. Therefore, to prove the convexity of F, it is sufficient to show $M(\theta)$ is convex. To accomplish this, we rewrite M as an iterated integral and show that the inner integral with respect to x is a convex function of θ.

We note that

$$M(\theta) = \int \left[\int [\nabla(\psi - cy)]^2 dx \right] dy .$$

Let us now define the functional

$$m(\theta,y) = \int [\nabla(\psi(\theta) - cy)]^2 \, dx ,$$

where $m(\theta,y)$ forms the integrand of $M(\theta)$, i.e.,

$$M(\theta) = \int m(\theta,y)dy .$$

We proceed now to establish the convexity of $m(\theta,y)$. The level curves of ψ are the curves $\psi(x,y) = t$, where the quantity

[illegible] varies from [illegible] to [illegible]. We make a change of variables by introducing t as an independent variable replacing x. The variable x will then depend on t and y, thus becoming a dependent variable.

Then we have

$$\psi_x(\theta)^2 dx = \frac{1}{\left|\frac{\partial X_j}{\partial t}\right|}\, dt$$

if we are on the jth branch of the curve $\psi(\theta) = t$. Note $j = 1,2$. Similarly,

$$c^2 dx = c^2 \left|\frac{\partial X_j}{\partial t}\right| dt$$

and

$$\psi_y(\theta)^2 dx = \frac{1}{\left|\frac{\partial X_j}{\partial t}\right|}\left(\frac{\partial X_j}{\partial y}\right)^2 dt \ .$$

Expanding the expression for $m(\theta, y)$ gives

$$m(\theta, y) = \int [\psi_x(\theta)^2 + \psi_y(\theta)^2 - 2c\psi_y(\theta) + c^2]\, dx \ .$$

Thus, by making the above substitutions and letting a prime denote partial differentiation with respect to y and a dot denote partial differentiation with respect to t, we get

$$m(\theta, y) = \sum_{j=1}^{2} \int \left[\frac{(X_j')^2 + 1}{|\dot{X}_j|} + c^2 |\dot{X}_j| - cX_j' \operatorname{sgn} \dot{X}_j\right] dt \ .$$

We now determine the signs of $\dot{X}_1$ and $\dot{X}_2$. Note that

$$\dot{X}_1 = X_1'\left(\frac{\partial y}{\partial t}\right)$$

and

$$\dot{X}_2 = X_2'\left(\frac{\partial y}{\partial t}\right) .$$

The quantity $\frac{\partial y}{\partial t}$ is always positive since t varies from 0 to $-\infty$ as we descend from the upper boundary of D to $y = -\infty$. Now, recall that

$$X_1 = x_1 - \theta\left(\frac{x_1 + x_2}{2}\right)$$

and (1)

$$X_2 = x_2 - \theta\left(\frac{x_1 + x_2}{2}\right)$$

where $0 \leq \theta \leq 2$. The branch $x_1(y)$ of $\psi = t$ was chosen so that it always has positive slope and the branch $x_2(y)$ was chosen so that it always has negative slope; that is

$$x_1'(y) > 0 \quad \text{and} \quad x_2'(y) < 0 .$$

We see, by differentiation of equations (1) with respect to y, that

$$X_1' = x_1' - \frac{\theta}{2}(x_1' + x_2')$$

and

$$X_2' = x_2' - \frac{\theta}{2}(x_1' + x_2')$$

where $0 \leq \theta \leq 2$. Hence we may write

$$X_1' = (1 - \frac{\theta}{2})x_1' - \frac{\theta}{2}x_2'$$

and

$$X_2' = (1 - \frac{\theta}{2})x_2' - \frac{\theta}{2}x_1' ,$$

where $[illegible] \geq 1 - \frac{\theta}{2} \geq 1$. Thus it is obvious that $X_1' > 0$ and $X_2' < 0$. This implies that

$$\dot{X}_1 > 0$$

and

$$\dot{X}_2 < 0 \ .$$

We rewrite m by substituting +1 for sgn $\dot{X}_1$ and -1 for sgn $\dot{X}_2$. We also rewrite $|\dot{X}_1|$ as $\dot{X}_1$ and $|\dot{X}_2|$ as $-\dot{X}_2$. Thus we have

$$m(\theta, y) = \int \left[\frac{(X_1')^2 + 1}{\dot{X}_1} + c^2 \dot{X}_1 - cX_1' \right] dt$$

$$+ \int \left[\frac{(X_2')^2 + 1}{-\dot{X}_2} - c^2 \dot{X}_2 + cX_2' \right] dt \ .$$

We now proceed to show the convexity of m with respect to θ by making use of the linearity of the transformation (1) in θ. Let

$$X_{1,1} = X_1(\theta_1) \ , \quad X_{1,2} = X_1(\theta_2)$$

$$X_{2,1} = X_2(\theta_1) \ , \quad X_{2,2} = X_2(\theta_2) \ .$$

Then

$$X_1 \left(\frac{\theta_1 + \theta_2}{2} \right) = \frac{1}{2} (X_{1,1} + X_{1,2})$$

and

$$X_2 \left(\frac{\theta_1 + \theta_2}{2} \right) = \frac{1}{2} (X_{2,1} + X_{2,2})$$

since we have linearity in θ. Substitution in the equation for $m(\theta)$ gives

$$m\left(\frac{\theta_1+\theta_2}{2}, y\right) = \sum_{j=1}^{2} (-1)^{j+1} \int \left[\frac{\left(\frac{X'_{j,1}+X'_{j,2}}{2}\right)^2 + 1}{\left(\frac{\dot{X}_{j,1}+\dot{X}_{j,2}}{2}\right)} + c^2\left(\frac{\dot{X}_{j,1}+\dot{X}_{j,2}}{2}\right) - c\left(\frac{X'_{j,1}+X'_{j,2}}{2}\right) \right] dt \; .$$

We may rewrite the non-linear terms in the above integral as

$$(-1)^{j+1}\left[\frac{\left(\frac{X'_{j,1}+X'_{j,2}}{2}\right)^2+1}{\left(\frac{\dot{X}_{j,1}+\dot{X}_{j,2}}{2}\right)}\right] = (-1)^{j+1}\left\{\frac{(X'_{j,1})^2+1}{2\dot{X}_{j,1}} + \frac{(X'_{j,2})^2+1}{2\dot{X}_{j,2}} - \left[\frac{(\dot{X}_{j,1}-\dot{X}_{j,2})^2 + (\dot{X}_{j,1}X'_{j,2} - X'_{j,1}\dot{X}_{j,2})^2}{2\dot{X}_{j,1}\dot{X}_{j,2}(\dot{X}_{j,1}+\dot{X}_{j,2})}\right]\right\} \; .$$

This is done by completing the square. Since the other terms in the integrand are linear in θ, we have

$$(2) \quad m\left(\frac{\theta_1+\theta_2}{2}, y\right) = \frac{1}{2}\, m(\theta_1, y) + \frac{1}{2}\, m(\theta_2, y) - \sum_{j=1}^{2} (-1)^{j+1} \int \left[\frac{(\dot{X}_{j,1}-\dot{X}_{j,2})^2 + (\dot{X}_{j,1}X'_{j,2} - X'_{j,1}\dot{X}_{j,2})^2}{2\dot{X}_{j,1}\dot{X}_{j,2}(\dot{X}_{j,1}+\dot{X}_{j,2})}\right] dt \; .$$

Thus

$$(3) \qquad m\left(\frac{\theta_1+\theta_2}{2}\right) \le \frac{1}{2}\,[m(\theta_1, y) + m(\theta_2, y)] \; ,$$

which means that $m(\theta, y)$ is convex. The method used in the derivation of this is equivalent to an application of Schwarz's inequality.

We are now ready to prove that $F(\theta)$ is well-defined by showing the convergence of $M(\theta)$. We note that

$$m(\theta,y) \geq 0 \ ,$$

since the integrand is always positive. From the convexity of $m(\theta, y)$, we may conclude that

$$m(\theta,y) \leq \max\,(m(0,y), m(2,y))$$

$$\leq m(0,y) + m(2,y) \ .$$

We recall that

$$M(\theta) = \int m(\theta,y)dy \ ,$$

and therefore

$$M\left(\frac{\theta_1+\theta_2}{2}\right) = \int m\left(\frac{\theta_1+\theta_2}{2}, y\right)dy$$

$$\leq \frac{1}{2}\left[\int m(\theta_1,y)dy + \int m(\theta_2,y)dy\right]$$

$$= \frac{1}{2}\,[M(\theta_1) + M(\theta_2)] \ ,$$

which means that $M(\theta)$ is convex. Hence, we have established the convexity of $F(\theta)$. Since $M(\theta) = \int m(\theta,y)dy$, we have

$$0 \leq M(\theta) \leq M(0) + M(2) \ .$$

Because $M(0)$ and $M(2)$ converge, we conclude that $M(\theta)$ must converge for $0 \leq \theta \leq 2$. Thus the functional

$$F(\theta) = M + (\lambda - c^2)A - E$$

is well-defined.

We may separate the formula for δF, the first variation of F, into two parts, a part arising from the variation $\delta\psi$ of the stream function ψ over the fixed domain D and a part corresponding to the normal variation δn of the boundary of the domain D. Thus

$$\delta F = \iint_D \delta[\nabla(\psi - cy)]^2 \, dA + \int_C (q^2 + 2gy - \lambda)\delta n ds \ ,$$

where the second term of the formula, representing the part from varying the domain, was derived in Chapter 2. For $\theta = 0$ and $\theta = 2$, the first term vanishes through an application of Dirichlet's principle, while the second term vanishes because $\psi(x,y)$ and $\psi(-x,y)$ satisfy Bernoulli's law

$$q^2 + 2gy = \lambda$$

on the free boundary, with the same constant λ. Thus

$$F'(0) = F'(2) = 0 \ .$$

We may conclude, therefore, that $F(\theta)$ is a convex function whose derivative $F'(\theta)$ vanishes at the end points of the interval $0 \leq \theta \leq 2$. Hence $F(\theta)$ must be constant in this interval.

[illegible]

$$\dot{X}_{\sigma,1} = \dot{X}_{\sigma,2}$$

for $i = 1,2$ and for all σ, where $0 \leq \sigma \leq 2$. In particular

$$\dot{X}_1(0) = \dot{X}_1(2) .$$

However,

$$X_1(0) = x_1$$

and

$$X_1(2) = -x_2 .$$

Thus

$$\dot{x}_1(y) = -\dot{x}_2(y) .$$

This gives us a contradiction unless $\psi(x,y) = t$ and $\psi(-x,y) = t$ define the same curves for all t. Hence the flow ψ has symmetry with respect to a vertical line through a crest, or, equivalently, through a trough.

3.II.a. A Theorem on the Behavior of Streamlines at Great Depth

Theorem 3.II.a. Let $\psi(x,y)$ be a periodic stream function having period one with respect to its x-coordinate and such that $(\nabla\psi)^2 = O(1)$ as $y \to -\infty$. Assume the curve $\psi(x,y) = 0$ ascends and descends just once in a period. Then the level curves $\psi(x,y) = t$ have the same property for t large and negative.

Proof: Let $\mathcal{G}$ represent one period of the wave (see Fig. 4). We note that the side boundaries of $\mathcal{G}$ are not necessarily

vertical as for the domain D described in previous sections. The upper boundary of $\mathcal{D}$, which we denote by C, will fall on the level curve $\psi = 0$.

Using the complex potential ζ, we map the domain $\mathcal{D}$ conformally onto the semi-infinite rectangle $\psi < 0$, $0 < \phi < c$ in the ζ-plane (see Fig. 4), where c (the speed at $y = -\infty$), represents the amount $\phi(x,y)$ increases as we traverse one period of the wave.

We map the rectangle plus the part of the boundary falling on the curves $\psi = 0$ and $\phi = 0$ onto the unit disc in the ζ_1-plane by the transformation

$$\zeta_1 = e^{-i\zeta\frac{2\pi}{c}} = e^{(\psi - i\phi)\frac{2\pi}{c}}$$

(see Fig. 5). The point $\psi = -\infty$ corresponds to the origin in the ζ_1-plane and the curve $\psi = 0$ for $0 \leq \phi < c$ maps one-to-one onto the unit circle $|\zeta_1| = 1$. Let

$$\omega = qe^{-i\theta} = \frac{d}{dz}(\phi + i\psi) .$$

The angle θ can then be interpreted as the angle of inclination of the velocity vector with respect to the positive x-axis.

We consider the image of the unit disc $|\xi_1| \leq 1$, in the plane of $\log \omega$ (see Fig. 5). Without loss of generality, we make the assumption that the speed at $y = -\infty$ is one. Then the origin in the ζ_1-plane maps onto the origin in the plane of $\log \omega$. Let us call the image of the unit circle $|\xi_1| = 1$, C".

[illegible]

$$q^2 + 2qy \; [illegible] \; .$$

and the assumption that [illegible] rises and falls exactly once to show that the mapping of C onto C" is one-to-one. For $\theta > 0$, y is monotonically increasing as we traverse C, which implies q is monotonically decreasing. Similarly, for $\theta < 0$, y is monotonically decreasing as we continue along C, which implies q is monotonically increasing. Thus we may write

$$\theta = \theta(\log q) \; ,$$

where this function has exactly two branches; one for $\theta > 0$ and one for $\theta < 0$. This means that the mapping of C onto C" is one-to-one. Since the inverse mapping of $|\zeta_1| = 1$ onto C is one-to-one, the map of $|\zeta_1| = 1$ onto C" is one-to-one.

Now we may apply the argument principle to conclude that any point $\log \omega_0$, inside the domain bounded by C" has exactly one pre-image in the interior of the unit circle in the ζ_1-plane. The pre-image of the real axis segments $\overline{\overline{OA}}$, $\overline{\overline{OB}}$ in the plane of $\log \omega$ are two curves $\overline{OA}$, $\overline{OB}$ in the ζ_1-plane intersecting at the origin.

We now show, for t sufficiently large and negative, that the image of a level curve $\psi = t$ in the plane of $\log \omega$ intersects $\theta = 0$ exactly twice. Since ω is analytic near $\zeta_1 = 0$, we may write

$$\omega = 1 + \sum_{n=1}^{\infty} a_n \zeta_1^n$$

in a neighborhood of $\zeta_1 = 0$. We note that a_1 cannot be zero because in that case the mapping would not be one-to-one in a neighborhood of $\zeta_1 = 0$. Since conformal maps take small circles into almost circular curves, for t large and negative, we may say that the image of a circle $\psi = t$ in the ζ_1-plane will be an almost circular curve in the plane of $\log \omega$. Hence the image of $\psi = t$ in the plane of $\log \omega$ will cut the line $\theta = 0$ exactly twice. This means the image curve can be divided into two arcs, one on which θ is positive and one on which θ is negative. We may now conclude that a level curve $\psi = t$ rises and falls exactly once in a period for t large and negative. This is what we set out to prove.

3.II.b. A Discussion on the Behavior of Streamlines of ψ

In this section we make an additional restriction on the upper boundary C of the domain G, which represents one period of the flow. This will enable us to obtain the result of the previous section for all the level curves of ψ, that is, if C rises and falls once, then for all t, $\psi = t$ rises and falls just once in one period. The additional restriction that we place on C is that its image in the plane of $\log \omega$, which we call C", bounds a starlike region with respect to the origin of that plane.

Since the region that C" bounds is starlike, the inner product of the normal to the curve C" and the radius vector is always ≥ 0. Since $\psi = 0$ on C" and $(\log q, \theta)$ is a radius vector,

We see that

$$(\log q)\,\frac{\partial\psi}{\partial \log q} + \theta\,\frac{\partial\psi}{\partial\theta} \geq 0\ .$$

We note that ψ is a Green's function on the domain bounded by C" and that

$$h = (\log q)\,\frac{\partial\psi}{\partial \log q} + \theta\,\frac{\partial\psi}{\partial\theta}$$

is harmonic in this domain. The latter holds because h is the imaginary part of an analytic function in this domain, except possibly at the origin. We note, however, that this singularity is removable.

Since $h \geq 0$ on C", it follows from the maximum principle that $h > 0$ inside. Thus, for any t, the image of a level curve $\psi = t$ is starlike with respect to the origin in the plane of log w. This means that any line through the origin in the plane of log w will intersect the image of $\psi = t$ twice. In particular, $\theta = 0$ cuts the image twice.

We conclude that $\psi = t$ can be divided into two segments, one on which θ is positive and one on which θ is negative. This means that θ rises and falls once in one period, which is the desired result.

. A Theorem on the Non-Parametric Representation of the Free Streamline

Through an application of Hopf's lemma for subharmonic functions [1], we will derive the theorem that the free streamline has a non-parametric representation $y = y(x)$, where

$y(x)$ is a single-valued function of x with no vertical tangent at any point. This means the wave cannot bend back on itself and thus is convex with respect to the y-direction.

In the next chapter we will discuss a variational principle which asserts that, for fixed A and E, M is a minimum when ψ describes a wave. If we restrict ourselves to one period of the wave, then by symmetrization we can prove that the free surface of the solution to the minimum problem should have a non-parametric representation $x = x_j(y)$ with just two branches which are symmetric with respect to the y-axis. These two theorems on the non-parametric representation of the free surface of the wave enable us to restrict the class of competing functions for the variational principle and thus contribute to an existence proof for the wave problem by variational methods.

We proceed to the proof of the theorem. As in previous sections, the domain D represents one period of the flow described by the stream function $\psi(x,y)$. For this theorem, we assume $\psi(x,y)$ describes a wave and hence satisfies Bernoulli's law, $q^2 + 2gy = \lambda$, on $\psi = 0$, the upper boundary of D.

Let

$$f(x,y) = q^2 + 2gy - \lambda$$

throughout the domain D, where $q^2 = \psi_x^2 + \psi_y^2$. We have

$$f(x,y) \to -\infty$$

as $y \to -\infty$, since q^2 is bounded and λ is a constant. Also, $f(x,y) \equiv 0$ along the free boundary C, and

$$\Delta f \geq$$

in the domain D; in other words, f is subharmonic. Hence, by the maximum principle for subharmonic functions,

$$f < 0$$

throughout D.

Let $\vec{n}$ denote the inner normal to the free boundary C and let θ denote the angle that the tangent to C makes with the positive x-axis. We let s denote the arc length parameter along C, where s = 0 at the left hand end point of C, and define the curvature κ by the formula

$$\kappa = \frac{d\theta}{ds} .$$

Differentiation of f with respect to $\vec{n}$ gives us

$$\frac{\partial f}{\partial n} = 2q \frac{\partial q}{\partial n} + 2g \frac{\partial y}{\partial n} .$$

Using the relation

$$\frac{\partial q}{\partial n} = -\kappa q ,$$

we have

$$\frac{\partial f}{\partial n} = -2q^2\kappa + 2g \frac{\partial y}{\partial n} .$$

We now state Hopf's lemma, which we will apply to the function $\frac{\partial f}{\partial n}$: Let D be a domain bounded by a smooth curve and let u be a non-positive subharmonic function defined on D, and having continuous first partial derivatives on the boundary of D. Let

$\vec{n}$ denote the inner normal along the boundary of D. If $u = 0$ at a point on the boundary of D, then $\frac{\partial u}{\partial n} \leq 0$ at that point. Unless $u \equiv 0$, we get strict inequality. Applying Hopf's lemma, we obtain

$$\frac{\partial f}{\partial n} < 0 .$$

Hence

$$2g \frac{\partial y}{\partial n} < \kappa q^2 .$$

This means that at any point along C where the tangent is vertical, the curvature must be strictly positive.

Suppose the curve C has a vertical tangent at some point P_1 along it. Then without loss of generality, we may take the normal at P_1 in the positive x-direction. Hence $\frac{\partial y}{\partial n} = 0$. This implies, via the above inequality, that κ must be positive at P_1. Since κ is continuous along C, κ must be positive in a neighborhood of P_1. The only way for this to happen is for the curve C to bend back on itself, i.e., it must be that if x_1 is the x-coordinate of P_1, then there is no single-valued function $y = y(x)$ describing C in a neighborhood of $x = x_1$. The curve C must therefore be concave toward the fluid in a neighborhood of P_1, i.e., locally the curve will lie to the left of $x = x_1$ (see Fig. 6).

Suppose we move a vertical line toward the curve C, from the left. Then this line must be tangent to C at a point $P_2(x_2,y_2)$ where $x_2 < x_1$. Note that if s_1 and s_2 are respectively the values of s corresponding to P_1 and P_2, then we may

[illegible] $> y_1$ [illegible]. We see that the [illegible]

[illegible]

[illegible] P. This means κ is negative in a neighborhood of P_2.

This contradicts the inequality

$$2r \frac{\partial y}{\partial n} < \kappa q^2$$

at P_2, which says that κ must be positive. Hence the curve C can have no vertical tangent at any point and has a non-parametric representation $y = y(x)$, where $y(x)$ is a single-valued function of x.

CHAPTER 4

THE SECOND VARIATION OF M

In this chapter we set up a variational principle for the wave. This principle asserts that if A and E are held fixed, then the stream function ψ describing the wave minimizes M. We recall that in Chapter 3 we showed how we could restrict the class of competing functions which would enter into an existence proof for waves by variational techniques. If we limited ourselves to one period of the free surface, then symmetrization indicated that the wave had the non-parametric representation $x = x_j(y)$ with just two branches, which are symmetric about a trough or crest, and Hopf's lemma implied that the wave had the non-parametric representation $y = y(x)$. Thus the class of admissible functions could be limited to functions having these two properties.

We note that only certain values of E will give a solution. This is because, if we fix A alone, then a minimum is furnished for E when we consider the flat case of uniform flow. Thus E must be larger than this value. If we allow the amplitude to get too large, then again we presumably get no solution because the wave must break.

We show in this chapter that the above variational principle is correct by deriving a formula for the second variation of M and showing it to be strictly positive when A and E are held fixed up to second order.

We have already shown in Chapter 2 that if ψ describes a wave then

$$\delta F = \delta(M + (1-c^2)A - E) = 0 .$$

If also

$$\delta A = 0$$

and

$$\delta E = 0$$

then clearly

$$\delta M = 0 .$$

Let us denote the second variation by δ^2. We will establish the following result.

<u>Theorem</u>. If

$$\delta A = \delta^2 A = 0$$

and

$$\delta E = \delta^2 E = 0$$

then not only $\delta M = 0$, but also

$$\delta^2 M > 0$$

provided ψ describes a wave. This will imply that the minimum problem is correct for waves. In this paper, however, we are able only to prove the theorem for waves of sufficiently small amplitude.

4.I.a. Derivation of the Second Variation of M

Let

$$J(\varepsilon) = \iint_D F(x,y,\psi,\psi_x,\psi_y)\,dA$$

be a functional which varies analytically with the parameter ε. By Taylor's theorem, we may write the series

$$J(\varepsilon) = J(0) + \left(\frac{dJ}{d\varepsilon}\right)\varepsilon + \frac{1}{2}\left(\frac{d^2J}{d\varepsilon^2}\right)\varepsilon^2 + \dots \; .$$

Then the first variation of J is defined as

$$\delta J = \left(\frac{dJ}{d\varepsilon}\right)\Big|_{\varepsilon=0} \varepsilon$$

and the second variation is defined as the next term in the series, that is,

$$\delta^2 J = \frac{1}{2}\left(\frac{d^2J}{d\varepsilon}\right)\Big|_{\varepsilon=0} \varepsilon^2 \; .$$

In Section 2.II.a., we derived the formula

$$\delta M = -\int_C (c^2 - q^2)\,\delta n\,ds$$

for the first variation of M. We took $\vec{n}$ to be the inner normal to the analytic curve C which forms the upper boundary of D and we let

$$\delta n = \varepsilon\rho(s)$$

where [illegible] is a real analytic function of the arc length parameter along C.

In this section we derive the following formula for $\delta^2 M$:

$$\delta^2 M = -\frac{1}{2}\int_C ([illegible]^2 + [illegible]^2)\kappa(\delta n)^2 ds + \iint_D (\nabla(\delta\psi))^2 dA \ .$$

where $\vec{n}$, C, and D have the same meanings as described above in the formulation of the expression for δM, and κ is the curvature along C. We let ψ be a stream function defined on D. As before

$$\psi = 0$$

on C,

$$\psi(0,y) = \psi(1,y)$$

for all y, and

$$(\nabla\psi)^2 = O(1)$$

as $y \to -\infty$.

We obtain the domain D^* by varying only the upper boundary of D. Let C^* be the upper boundary of D^* and let ψ^* be the corresponding stream function defined on D^*, which has the following properties:

$$\Delta\psi^* = 0$$

throughout D^*,

$$\psi^*(1,y) = \psi^*(0,y)$$

for all y, and

$$(\nabla\psi^*)^2 = O(1)$$

as $y \to -\infty$.

We make the assumption that C and C^* have the same initial and terminal points. Without loss of generality the formula for the second variation of M will then be derived under the additional hypothesis that C and C^* intersect each other once between their common end points (see Fig. 7). The techniques involved in proving the formula for the case of one intersection are easily extended to an arbitrary number of intersections.

Let $C = C_1 + C_2$ and $C^* = C_1^* + C_2^*$, where C_1 and C_1^* bound $D-D^*$ and C_2 and C_2^* bound D^*-D. Since C is an analytic curve and $\rho(s)$ is a real analytic function, C^* is also analytic. This implies, by an application of the Schwarz reflection principle, that ψ and ψ^* can be extended outside of their original domains of definition in such a way so that they remain harmonic. Hence we can take C and C^* so close that ψ and ψ^* are harmonic in $D-D^*$ and D^*-D.

Let

$$\delta\psi = \psi^* - \psi \; .$$

Then $\delta\psi$ is harmonic in $(D-D^*) \cup (D^*-D) \cup (D \cap D^*)$, and the normal derivative of $\delta\psi$ vanishes on the vertical boundaries of D and D^*. We will need the expansion of $\delta\psi$ for y large and negative. From Theorem 2.I.a., we have

$$\psi = y_o + cy + \sum_{n=1}^{\infty} a_n e^{2\pi n y} \cos\,(2\pi n x + \phi_n)$$

and

$$\psi^* = y^* + \ldots + \sum_{n=1}^{\infty} a_n^* e^{2\pi n y} \cos\,(2\pi n x + \phi_n^*)$$

near $y = -\infty$. Thus

$$\delta\psi = y^* - y + \sum_{n=1}^{\infty} b_n e^{2\pi n y} \cos\,(2\pi n x + \tilde{\phi}_n) \; .$$

From this expansion, we see that

$$\left| \frac{\partial(\delta\psi)}{\partial y} \right| \leq K e^{2\pi y}$$

for y large and negative. The expansion for $\delta\psi$ also implies that the Dirichlet integrals for $\delta\psi$ over D and D^* both converge.

In Section 2.II.a.. we saw that we may write

$$(M^*-M) - c^2(A^*-A) = \iint\limits_{D-D^*} \nabla\psi \cdot \nabla\psi^* dA - \iint\limits_{D^*-D} \nabla\psi \cdot \nabla\psi^* dA \; .$$

Since $\psi^* = \psi + \delta\psi$,

$$\iint\limits_{D-D^*} \nabla\psi \cdot \nabla\psi^* dA - \iint\limits_{D^*-D} \nabla\psi \cdot \nabla\psi^* dA = \iint\limits_{D-D^*} \nabla\psi \cdot \nabla(\psi + \delta\psi) dA$$

$$- \iint\limits_{D^*-D} \nabla\psi \cdot \nabla(\psi + \delta\psi) dA \; .$$

Substituting $(M^*-M) - c^2(A^*-A)$ for the right hand side of the equation and expanding the left hand side gives

$$(M^*-M) - c^2(A^*-A) = \iint_{D-D^*} (\nabla\psi)^2 dA + \iint_{D-D^*} \nabla\psi \cdot \nabla(\delta\psi) dA$$

$$+ \iint_{D-D^*} (\nabla(\delta\psi))^2 dA - \left(\iint_{D^*-D} (\nabla\psi)^2 dA \right.$$

$$\left. + \iint_{D^*-D} \nabla\psi \cdot \nabla\delta\psi dA + \iint_{D^*-D} (\nabla(\delta\psi))^2 dA \right) .$$

An application of Green's theorem to the domains $D-D^*$ and D^*-D, using inward normals along the boundary curves with respect to the domains D and D^*, respectively, gives

$$(M^*-M) - c^2(A^*-A) = \iint_{D-D^*} (\nabla\psi)^2 dA + \int_{C_1^*} \psi \frac{\partial}{\partial n} (\delta\psi) ds$$

$$- \int_{C_1} \psi \frac{\partial}{\partial n} (\delta\psi) ds - \left(\iint_{D^*-D} (\nabla\psi)^2 dA \right.$$

$$\left. - \int_{C_2^*} \psi \frac{\partial(\delta\psi)}{\partial n} ds + \int_{C_2} \psi \frac{\partial(\delta\psi)}{\partial n} ds \right) .$$

Since ψ is zero along C and hence along C_1 and C_2, and since $\psi = -\delta\psi$ on C^*, by combining the two non-zero line integrals, we have

$$(M^*-M) - c^2(A^*-A) = \iint_{D-D^*} (\nabla\psi)^2 dA - \iint_{D^*-D} (\nabla\psi)^2 dA - \int_{C^*} (\delta\psi) \frac{\partial}{\partial n} (\delta\psi) ds .$$

Let D_k^* be the domain which has as its upper boundary the curve C^*, as its lower boundary the line $y = -k$ where $k > 0$, and whose side boundaries are the usual vertical lines. Then

$k \to \infty$. An application of Green's theorem to the integral $-\int_{\Gamma'} (\delta\psi) \frac{\partial}{\partial n} (\delta\psi) ds$ gives

$$-\int_{\Gamma'} (\delta\psi) \frac{\partial}{\partial n} (\delta\psi) ds = \iint_{D'_k} [\nabla(\delta\psi)]^2 dA + \int_{x=0}^{-k} (\delta\psi) \frac{\partial}{\partial n} (\delta\psi) ds$$

$$+ \int_{x=1}^{-k} (\delta\psi) \frac{\partial}{\partial n} (\delta\psi) ds + \int_{y=-k}^{1} (\delta\psi) \frac{\partial}{\partial n} (\delta\psi) ds$$

for every $k > 0$.

Since we know that $\delta\psi$ is periodic, the second and third integrals on the right hand side of the equation cancel each other. It was shown above that $\frac{\partial}{\partial n} (\delta\psi) \to 0$ exponentially as $k \to \infty$. This implies

$$\lim_{k \to \infty} \int_{y=-k}^{1} (\delta\psi) \frac{\partial}{\partial n} (\delta\psi) ds = 0 .$$

Thus we have, by taking the limit as $k \to \infty$,

$$-\int_{\Gamma'} (\delta\psi) \frac{\partial}{\partial n} (\delta\psi) ds = \iint_{D'} [\nabla(\delta\psi)]^2 dA .$$

Hence

$$(K'-K) - c^2(A'-A) = \iint_{D-D'} (\nabla\psi)^2 dA - \iint_{D'-D} (\nabla\psi)^2 dA + \iint_{D'} [\nabla(\delta\psi)]^2 dA .$$

Replacing the integral over D' with an integral over D gives

$$(M^*-M) - c^2(A^*-A) = \iint_{D-D^*} (\nabla\psi)^2 dA - \iint_{D^*-D} (\nabla\psi)^2 dA$$

$$+ \iint_D [\nabla(\delta\psi)]^2 dA + O(\varepsilon^3) .$$

We now express dA in terms of normal and tangential coordinates and replace $(\nabla\psi)^2$ by q^2 to get

$$(M^*-M) - c^2(A^*-A) = \iint_{(D-D^*)-(D^*-D)} q^2 \frac{\partial(x,y)}{\partial(n,s)} \, dnds$$

$$+ \iint_D [\nabla(\delta\psi)]^2 dA + O(\varepsilon^3) .$$

Inserting an expression for the Jacobian valid up to second order terms and writing q^2 in terms of its value on the curve C through Taylor's theorem, we get

$$(M^*-M) - c^2(A^*-A) = \iint_{(D-D^*)-(D^*-D)} \left(q^2 + \frac{\partial q^2}{\partial n} n\right)(1+\kappa n) dnds$$

$$+ \iint_D (\nabla(\delta\psi))^2 dA + O(\varepsilon^3) .$$

Here κ denotes the curvature on the curve C and q^2 denotes the value of q^2 along C. Integrating with respect to the normal coordinate and dropping off terms of higher than second order gives

$$(M^{\bullet}-M) - c^2(A^{\bullet}-A) = \int_C q^2(\delta n)ds + \int_C \frac{\partial q^2}{\partial n}\frac{(\delta n)^2}{2}ds$$

$$+ \int_C \frac{\kappa}{2} q^2(\delta n)^2 ds + \iint_D [\nabla(\delta\psi)]^2 dA + O(\varepsilon^3) .$$

Since up to second order

$$A^{\bullet}-A = -\int_C (1 + \frac{\kappa}{2}\delta n)(\delta n)ds$$

and since $\frac{\partial q^2}{\partial n} = -2\kappa q^2$, we get the following result:

$$M^{\bullet}-M = -\int_C (c^2 - q^2)\delta n ds - \frac{1}{2}\int_C (c^2 + q^2)\kappa(\delta n)^2 ds$$

$$+ \iint_D [\nabla(\delta\psi)]^2 dA + O(\varepsilon^3) .$$

Because the second variation of M consists of all the above terms that are of second order in ε, we have

$$\delta^2 M = -\frac{1}{2}\int_C (c^2 + q^2)\kappa(\delta n)^2 ds + \iint_D [\nabla(\delta\psi)]^2 dA .$$

4.1.4. <u>The Second Variation of M For a Wave</u>

In Chapter 2, we showed that if the area A and the potential energy E were held fixed, and if ψ described a wave, then the first variation of M vanished. We now investigate the second variation of M.

$$\delta^2 M = -\frac{1}{2}\int_C (c^2+q^2)\kappa(\delta n)^2 ds + \iint_D [\nabla(\delta\psi)]^2 dA ,$$

when these conditions are imposed.

If A and E are held fixed, then we may say that

$$\delta A + \delta^2 A = 0$$

and

$$\delta E + \delta^2 E = 0$$

up to second order in ε. It is easily seen that we may write

$$\delta A + \delta^2 A = -\int_C (1 + \frac{\kappa}{2}\,\delta n)\delta n ds$$

and

$$\delta E + \delta^2 E = -\int_C 2gy\ \delta n ds - \int_C (gy\kappa + g\,\frac{\partial y}{\partial n})(\delta n)^2 ds .$$

Consider the following sum

$$\delta^2 M = (\delta M + \delta^2 M) + (\lambda - c^2)(\delta A + \delta^2 A) - (\delta E + \delta^2 E)$$

$$= \int_C (-\lambda + q^2 + 2gy)\delta n ds + \int_C \frac{1}{2}\,(-\lambda + q^2 + 2gy)\kappa(\delta n)^2 ds$$

$$+ \int_C (\frac{1}{2}\,\frac{\partial q^2}{\partial n} + g\,\frac{\partial y}{\partial n})(\delta n)^2 ds + \iint_D [\nabla(\delta\psi)]^2 dA + O(\varepsilon^3) .$$

Since ψ is a wave, it satisfies Bernoulli's law on C, i.e.,

$$q^2 + 2gy = \lambda .$$

This shows that the first and second integrals in the above equation vanish. Thus we arrive at the formula

$$\delta^2 M = \frac{1}{2}\int_C \frac{\partial}{\partial n}(q^2 + 2gy)(\delta n)^2 ds + \iint_D [\nabla(\delta\psi)]^2 dA .$$

By Hopf's lemma for subharmonic functions

$$\frac{\partial}{\partial n}(q^2 + 2gy) < 0 .$$

This implies that the first integral in the formula for $\delta^2 M$ is always negative. Note that the second integral is always positive.

[illegible].II. Investigation of the Sign of the Second Variation of M

We now have an expression for $\delta^2 M$ in the case of a wave when variations are made holding A and E fixed. This expression is

$$\delta^2 M = \frac{1}{2}\int_C \frac{\partial}{\partial n}(q^2 + 2gy)(\delta n)^2 ds + \iint_D [\nabla(\delta\psi)]^2 dA .$$

where the first term is negative and the second term is positive.

We want to show

$$\delta^2 M > 0$$

subject to two linear constraints on δn,

$$\delta A = 0 , \qquad \delta E = 0 .$$

We note that showing the above inequality is equivalent to showing that the quotient

$$\frac{\iint_D [\nabla(\delta\psi)]^2 dA}{-\frac{1}{2}\int_C \frac{\partial}{\partial n}(q^2 + 2gy)(\delta n)^2 ds} > 1$$

subject to the constraints

$$\delta A = -\int \delta n ds = 0$$

and

$$\delta E = -\int y \; \delta n ds = 0 \; .$$

Once we have proved this, we may conclude that the wave minimizes M for fixed A and E and the variational principle we stated for the wave is correct.

First we will consider the corresponding expression $\delta^2 M$ for the flat case of uniform flow and show it to be strictly positive. Then we will deduce from this result that $\delta^2 M$ is strictly positive for the wave provided we assume that we are close to the flat case.

4.II.a. The Sign of the Second Variation of M for the Flat Case

Recall that for a wave we have

$$\delta^2 M = \frac{1}{2}\int_C \left(\frac{\partial q^2}{\partial n} + 2g\,\frac{\partial y}{\partial n}\right)(\delta n)^2 ds + \iint_D [\nabla(\delta\psi)]^2 dA$$

[illegible]

[illegible]

[illegible], we may rewrite the above expression for $\delta^2 W$ in the form

$$\delta^2 W = \int_0^1 g(\delta n)^2 dx + \iint_R [\nabla(\delta\psi)]^2 dA \ ,$$

where R is the semi-infinite rectangle $0 \leq x \leq 1$, $y \leq$ [illegible]. Through an application of Taylor's theorem and noting that $q = \left(\frac{\partial\psi}{\partial n}\right)$ on [illegible], we may write

$$(\delta n)^2 = \frac{1}{q^2} (\delta\psi)^2 = \frac{1}{c^2} (\delta\psi)^2 \ .$$

By substituting this result for $(\delta n)^2$, we see that showing $\delta^2 W >$ [illegible] is equivalent to proving that

$$\frac{\iint [\nabla(\delta\psi)]^2 dA}{\int_0^1 \frac{g}{c^2} (\delta\psi)^2 dx} > 1$$

subject to the two constraints that A and E are held fixed.

In order to evaluate g/c^2 and to investigate our constraints, we consider the flat case as the limit of the classical linearized solution in which the amplitude h is small. We [illegible] linearized case briefly (cf. [illegible]).

Let $\Phi(x,y)$ be the velocity potential. For the linearized case we may redefine Φ by the expression

$$\Phi(x,y) = cx + \phi(x,y) ,$$

where the gradient of ϕ will be infinitesimally small. We may write

$$\nabla\Phi = (c,0) + (\phi_x,\phi_y) ,$$

where (ϕ_x,ϕ_y) exhibits the velocity due to the wave motion.

We now let $y = \eta(x)$ be the equation of the free surface of the wave, where $\eta(x)$ will be as small as we wish. We substitute $\eta(x)$ and $(\nabla\Phi)^2$ into Bernoulli's equation to get

$$H = g\eta + \frac{(\nabla\Phi)^2}{2} ,$$

where H is a constant. Expanding, we obtain

$$H = g\eta + \frac{c^2 + 2c\phi_x + \phi_x^2 + \phi_y^2}{2} ,$$

where the terms on the right hand side of the equation are evaluated at $y = \eta(x)$.

Since η and grad ϕ are small, we may linearize the above by simply dropping the quadratic terms and expanding in a Taylor's series about $y = 0$. Thus, applying Taylor's theorem to Bernoulli's equation, we have

$$\eta = \frac{H}{g} - \frac{c^2}{2g} - \left.\frac{c\phi_x}{g}\right|_{y=0} .$$

Now we establish the kinematic condition for the free surface by taking the time derivative of the equation $y - \eta(x) = 0$ and setting it equal to zero, which gives

$$\Phi_y - \eta_x \Phi_x = 0 \ .$$

Linearizing the equation gives

$$\phi_y - c\eta_x = 0 \ ,$$

or

$$\eta_x = \frac{\phi_y}{c} \ .$$

From the equation for $\eta(x)$ which was derived from Bernoulli's law, we have

$$\eta_x = - \frac{c}{g} \phi_{xx} \ .$$

Combining then two equations for η_x, we get

$$\eta_x = - \frac{c}{g} \phi_{xx} = \frac{\phi_y}{c} \ ,$$

which implies the classical result

$$\phi_y = - \frac{c^2}{g} \phi_{xx} \ .$$

The linearized equation just derived has the family of solutions

$$\phi = \hat{A} \, e^{ky} \sin kx \ .$$

where $k = g/c^2$. Our periodic boundary restriction gives us $k = 2\pi n$, and hence

$$2\pi n = \frac{g}{c^2} .$$

We shall use this relationship in the expression for $\delta^2 M$ in the flat case.

We combine the equations for ϕ and make the following substitution to display the small magnitude of the wave:

$$h = \frac{H - \frac{c^2}{2}}{g} .$$

We have

$$\eta = h[1 - \cos 2\pi nx]$$

and

$$\phi = ch\, e^{2\pi ny} \cos 2\pi nx .$$

We further restrict ourselves by taking the solution which has period one. Then

$$\eta = h[1 - \cos 2\pi x] ,$$

$$\psi = cy + ch\, e^{2\pi y} \cos 2\pi x ,$$

and

$$\frac{g}{c^2} = 2\pi .$$

We may now write down the expression for $\delta^2 M$ for the flat case by considering it as the limit as $h \to 0$ of the linearized solution of period one. This is

$$\delta^2 W = -2\pi \int_0^1 (\delta\psi)^2 dx + \iint_R [\nabla(\delta\psi)]^2 dA \ .$$

provided A and E are held fixed. Thus we wish to show that

$$\frac{\displaystyle\iint_R [\nabla(\delta\psi)]^2 dA}{\displaystyle 2\pi \int_0^1 (\delta\psi)^2 dx} > 1$$

subject to two linear constraints on $\delta\psi$.

We must investigate the variational constraints. As mentioned before, we assume

$$A'-A = \int_0^1 \delta n dx = 0 \ ,$$

where

$$\delta n = -\frac{1}{c}(\delta\psi) \ .$$

Thus

$$\int_0^1 \delta\psi dx = 0 \ .$$

Similarly

$$\delta E = -\int y \, \delta n \, dx \ .$$

To derive from this a nontrivial constraint in the case of uniform flow, we consider the limit of the linearized wave with free boundary

$$\eta = a(1 - \cos 2\pi x) \ .$$

We may write

$$-\int_0^1 h(1 - \cos 2\pi x)\delta\psi dx = 0$$

for each h. Using the earlier constraint on A, we conclude that

$$\int_0^1 (\cos 2\pi x)\delta\psi dx = 0 .$$

Our problem becomes to minimize the quotient

$$I = \frac{\iint_R (\nabla u)^2 dA}{\int_0^1 2\pi u^2 dx \Big|_{y=0}}$$

over the class S of functions u defined over the semi-infinite rectangle R such that

$$\Delta u = 0$$

throughout R,

$$u(0,y) = u(1,y)$$

for all $y \leq 0$, and $u \to 0$ exponentially as $y \to -\infty$. As a result we are led to the eigenvalue condition

$$\frac{\partial u}{\partial n} + 2\pi\lambda u = 0$$

[illegible] (0,1) [illegible] the x-axis. The corresponding [illegible] eigenfunctions are

$$u_0 = \frac{1}{\sqrt{2\pi}}$$

and

$$u_\lambda = \frac{1}{\sqrt{\pi}} e^{ky} \cos kx \; ,$$

where $k = 2\pi\lambda$ and $\lambda = 1,2,\ldots$. Thus zero is the lowest Stekloff eigenvalue and the remaining λ's are the other Stekloff eigenvalues. These are related to the linearized waves discussed earlier.

The eigenfunction u_0 gives the minimum of

$$I = \frac{\iint\limits_R (\nabla u)^2 dA}{\int_0^1 2\pi u^2 dx}$$

over all functions in the class S. This minimum value is of course zero. The eigenfunction u_1 minimizes the above ratio over all functions in S orthogonal to u_0 in the sense

$$\int_0^1 u\,dx = 0 \; .$$

We note that u_1 satisfies this orthogonality requirement because

$$\int_0^1 u_1\,dx = 0 \; .$$

We have

$$\frac{\iint_R (\nabla u_1)^2 dA}{\int_0^1 2\pi u_1^2 dx} = 1 ,$$

and thus

$$\frac{\iint_R [\nabla(\delta\psi)]^2 dA}{\int_0^1 2\pi(\delta\psi)^2 dx} \geq 1 .$$

To get the next eigenvalue, we minimize the above rati. with respect to all functions in S orthogonal to both u_o and u_1. A function in this class has the following properties:

$$\int_0^1 u dx = 0$$

and

$$\int_0^1 u \cos 2\pi x \, dx = 0 .$$

Since the next eigenvalue is $\lambda = 2$, for all such functions we have

$$\frac{\iint_R (\nabla u)^2 dA}{\int_0^1 2\pi u^2 dx} \geq 2 .$$

Because

$$\int_0^1 \delta\psi \cos 2\pi x dx = 0 ,$$

[illegible]

$$\frac{\iint_h [\,[illegible]\,]^2 \, dA}{\int^1 2\pi ([illegible])^2 \, dx} \geq 2 > 1 \ ,$$

which is what we set out to prove.

Our analysis shows that in the flat case of uniform flow we have

$$\delta^2 M \geq 0$$

under the single constraint $\delta A = 0$; but when both constraints $\delta A = \delta E = 0$ are imposed we obtain the stronger inequality

$$\delta^2 M > 0 \ .$$

The latter condition will enable us in the next section to estimate $\delta^2 M$ for waves of small finite amplitude.

4.II.b. The Sign of the Second Variation of M for Small Amplitude Waves

Let us assume that our wave is close to the solution for the flat case described in the previous section (4.II.a.). The stream function ψ may be expanded in a power series in the amplitude h which approaches the flat solution as $h \to 0$. We have, therefore,

$$\psi = \psi_0 + h\psi_1 + h^2\psi_2 + \ldots$$

for h sufficiently small. In this situation we will show that

$$\delta^2 M > 0$$

provided A and E are held fixed. The result implies that the wave minimizes M for fixed A and E.

We have the expression

$$\delta^2 M = -\int_C \rho(\delta\psi)^2 ds + \iint_D [\nabla(\delta\psi)]^2 dA \ ,$$

where

$$\rho = -\frac{1}{2}\frac{1}{q^2}\frac{\partial}{\partial n}(q^2 + 2gy)$$

and

$$(\delta\psi)^2 = q^2(\delta n)^2 \ .$$

We note that $\rho > 0$. The expression for $\delta^2 M$ will be positive provided we can show that

$$\frac{\iint_D [\nabla(\delta\psi)]^2 dA}{\int_C \rho(\delta\psi)^2 ds} > 1$$

when two linear constraints arising from fixing A and E are imposed on $\delta\psi$.

Let us consider these constraints. The constraint on A implies

$$\delta A = -\int_C \delta n ds = -\int_C \frac{1}{q}\,\delta\psi ds = 0 \ ,$$

which implies

$$0 = \frac{1}{h}\delta S = \int_{y=0}^{1} \frac{\delta\psi}{c}\,dx + O(h)$$

or

$$\int_{y=0}^{1} \delta\psi\,dx = O(h) .$$

The constraint on E can be written as

$$\delta E = -\int_C y\ \delta n\,ds = 0 .$$

Combining these constraints gives

$$0 = \frac{1}{h}\int_C y\ \delta n\,ds = \frac{1}{c}\int_{y=0}^{1} (\cos 2\pi x)\delta\psi\,dx + O(h) ,$$

which implies

$$\int_{y=0}^{1} (\cos 2\pi x)\delta\psi\,dx = O(h) .$$

Let us introduce the Nuemann's function so that we may replace the quotient

$$\tilde{I} = \frac{\iint_D (\nabla u)^2 dA}{\int_C \rho u^2 ds}$$

by an equivalent quotient of two quadratic forms defined over

the same space. The Neumann's function $N(z,\zeta)$ with respect to a domain D is a normalized fundamental solution of Laplace's equation. There are several ways to introduce the Neumann's function. For the application that we have in mind it is convenient to define the normal derivative in the following way: On C, we take

$$\frac{\partial N}{\partial n}(z,\zeta) = \frac{2\pi\alpha}{q} ,$$

where

$$\int_C \frac{1}{q}\, ds = \frac{1}{\alpha} ,$$

and on the vertical boundaries of D

$$\frac{\partial N}{\partial n} = 0 .$$

This choice of normal derivative is compatible with the condition

$$\frac{1}{2\pi}\int_{\partial D} \frac{\partial N}{\partial n}(z,\zeta)ds = 1$$

which holds because of the pole. For harmonic functions u which satisfy the condition

$$\int_C \frac{u}{q}\, ds = 0$$

this choice of $\frac{\partial N}{\partial n}$ simplifies the representation we derive for u in terms of its normal derivative on ∂D. The condition on the area

$$\int_C \frac{u}{q}\, ds$$ [illegible]

[illegible] a function. In particular, we may write the [illegible] application of Green's theorem

$$u(\zeta) = \frac{1}{2\pi}\int_{\partial D}\left[u(z)\,\frac{\partial N}{\partial n}(z,\zeta) - N(z,\zeta)\,\frac{\partial u}{\partial n}(z)\right]ds$$

$$= -\frac{1}{2\pi}\int_{\partial D} N(z,\zeta)\,\frac{\partial u}{\partial n}\, ds + \int_C \frac{\partial u}{q}\, ds\ .$$

This reduces to

$$u(\zeta) = -\frac{1}{2\pi}\int_C N(z,\zeta)\,\frac{\partial u}{\partial n}\, ds$$

provided we restrict ourselves to the class S' of harmonic functions u which have zero normal derivative along the vertical boundaries of D and such that

$$\int_C \frac{u}{q}\, ds = 0\ .$$

We shall also assume that functions in S' decay exponentially [illegible] as $y \to -\infty$.

Using the Neumann function $N(\zeta,z)$ for the domain D, we may rewrite the [illegible]

$$\tilde{I} = \frac{\iint_D (\nabla u)^2\, dA}{\int_C \rho u^2\, ds} - \frac{\int_C u\,\frac{\partial u}{\partial n}\, ds}{\int_C \rho u^2\, ds}$$

by applying Green's theorem and using the above representation for functions u in S'. We then get

$$\tilde{I} = \frac{2\pi \int\limits_C \int\limits_C N(\zeta,z) \frac{\partial u}{\partial n}(z) \frac{\partial u}{\partial n}(\zeta) ds dt}{\int\limits_C \int\limits_C \int\limits_C \rho N(\zeta,t) N(t,z) \frac{\partial u}{\partial n}(z) \frac{\partial u}{\partial n}(\zeta) ds dt d\sigma} .$$

We introduce a Hilbert space S^* which is a subspace of the spa[illegible] L_2 of Lebesgue square integrable functions. The subspace is defined by the orthogonality condition

$$\int\limits_C \int\limits_C \frac{N(z,\zeta)}{q} w(z) ds dt = 0$$

for all w(z) in S^*. This condition is equivalent to

$$\int\limits_C \frac{\delta\psi}{q} ds = 0$$

when

$$w(z) = \frac{\partial}{\partial n}(\delta\psi) .$$

Let

$$K(\zeta,z) = \int\limits_C \rho N(\zeta,t) N(t,z) dt .$$

Then minimizing $\tilde{I}$ over functions in S' is equivalent to minimizing the quotient Q_1/Q_2 of the two quadratic forms

$$Q_1 = \int\limits_C \int\limits_C N(\zeta,z) w(z) w(\zeta) ds dt$$

and

$$\mu_2 \; [illegible] \int_C\int_C K(\zeta,z)w(z)w(\zeta)\,ds\,dt$$

over the functions in S'.

We now proceed to show the desired inequality $t^2 M >$ [illegible] for fixed A and E by comparing

$$\tilde{I} = \frac{\int_C\int_C N(\zeta,z)w(\zeta)w(z)\,ds\,dt}{\int_C\int_C K(\zeta,z)w(z)w(\zeta)\,ds\,dt}$$

with the corresponding expression in the flat case of uniform flow. Specifically we show that for h sufficiently small and $0 < \delta < 1$

$$\frac{\int_C\int_C N(\zeta,z)w(z)w(\zeta)\,ds\,dt}{\int_C\int_C K(\zeta,z)w(z)w(\zeta)\,ds\,dt} \geq 2 - \delta > 1$$

for functions w in S' which satisfy the following condition arising from fixing A and E:

$$\int_C\int_C K(\zeta,z)w(z)u_1(\zeta)\,ds\,dt = O(h) ,$$

where $u_1 = e^{2\pi y}\cos 2\pi x$ is the first Stekloff eigenfunction discussed in 4.II.a. This inequality involving the Neumann's function will give us

$$\delta^2 M > 0$$

for fixed A and E.

For the flat case it is easily seen that for functions in S', the quotient I may be rewritten in the form

$$I = \frac{\int_0^1 \int_0^1 N_o(x,\zeta) \frac{\partial u}{\partial n}(\zeta) \frac{\partial u}{\partial n}(x) ds dx}{\int_0^1 \int_0^1 \int_0^1 2\pi [N_o(t,x) N_o(t,\zeta) \frac{\partial u}{\partial n}(\zeta) \frac{\partial u}{\partial n}(x)] dt ds dx}$$

where $N_o(x,\zeta)$ is the Neumann's function for the semi-infinite rectangle R. Then except for the eigenfunction $u_o = 1/\sqrt{2\pi}$, the Stekloff eigenfunctions described in Section 4.II.a., when evaluated at $y = 0$, are also the eigenfunctions for the above quotient I of quadratic forms. The eigenvalues are unchanged. Thus we have

$$\frac{\int_0^1 \int_0^1 N_o(x,\zeta) w(x) w(\zeta) dx ds}{\int_0^1 \int_0^1 K_o(x,\zeta) w(x) w(\zeta) dx ds} = I \geq 1$$

where $w(x)$ is in S^* and

$$K_o(x,\zeta) = \int_0^1 2\pi N_o(x,t) N_o(t,s) dt \ .$$

The orthogonality condition

$$\int_0^1 u \cos 2\pi x \, dx = 0$$

translates through the equality

$$\int_0^1 u \cos 2\pi x \, dx = \int_0^1 \int_0^1 K_0(x,t) \frac{\partial u}{\partial n}(t) \cos 2\pi x \, dx dt$$

into the constraint

$$\int_0^1 \int_0^1 K_0(x,t) w(t) \cos 2\pi x \, dx dt = 0 .$$

Imposing this constraint will therefore [cf. 4.II.a.] give us

$$\frac{\int_0^1 \int_0^1 H_0(x,\zeta) w(x) w(\zeta) dx ds}{\int_0^1 \int_0^1 K_0(x,\zeta) w(x) w(\zeta) dx ds} \geq 2 .$$

We note that the constraints for the non-linear case differ from the constraints imposed in the flat case of uniform flow [illegible] (h). Since [illegible] 2 [illegible] (h) and since the Neumann's function [illegible] the kernels of the two quadratic forms Q_1, Q_2 [illegible] [illegible] with the domain, we see that for h sufficiently small

$$\frac{\int_C \int_C H(\zeta,z) w(z) w(\zeta) ds dt}{\int_C \int_C K(\zeta,z) w(z) w(\zeta) ds dt} \geq 2 - \delta > 1 ,$$

where $\delta < 1$. Since $\frac{\partial}{\partial n}(\delta\psi)$ is in S^* and satisfies the constraint

$$\int_C \int_C K(z,\zeta) w(z) u_1(\zeta) ds dt = O(h) ,$$

we may assert that the above inequality holds for $w = \frac{\partial}{\partial n}(\delta\psi)$. Hence we conclude that

$$\delta^2 M > 0$$

for a wave provided the amplitude is sufficiently small, and A and E are held fixed.

Figure 1

z-plane

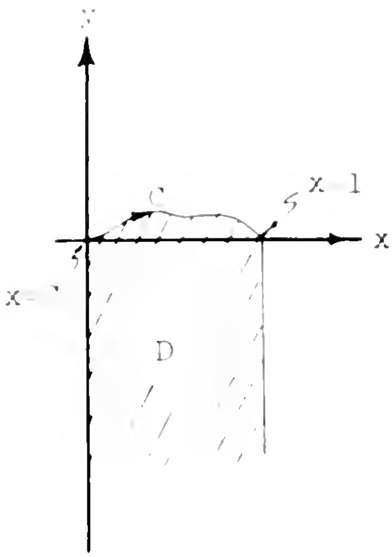


Figure 2

w-plane

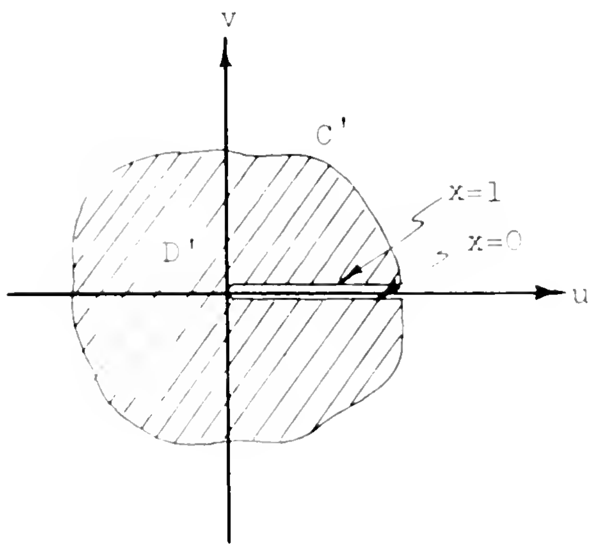


Figure 3

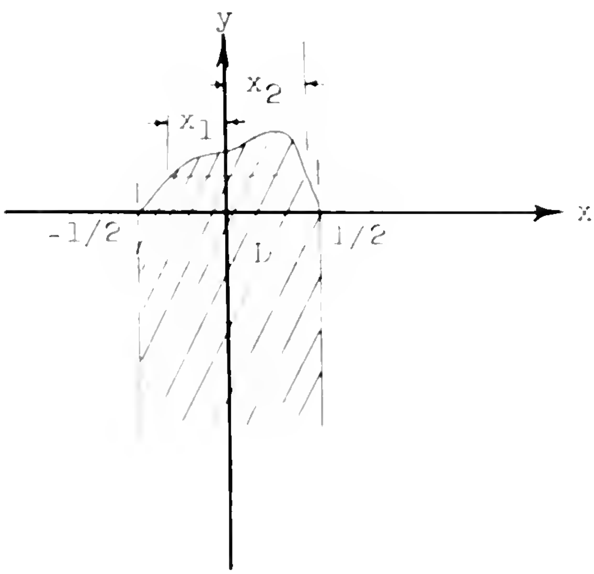
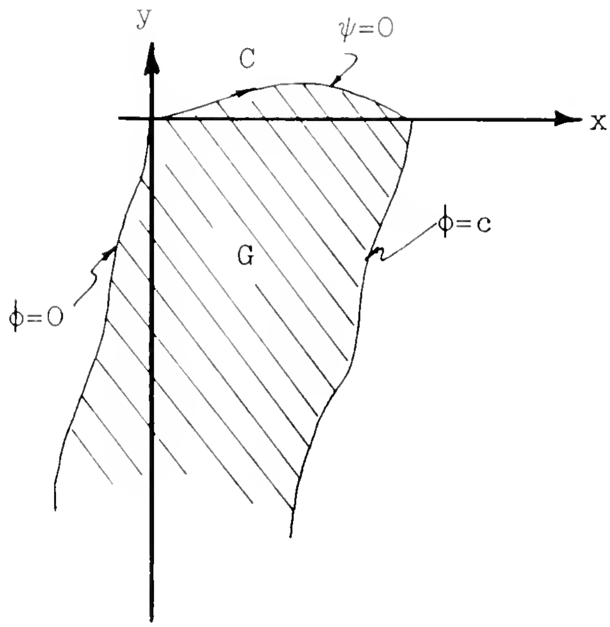
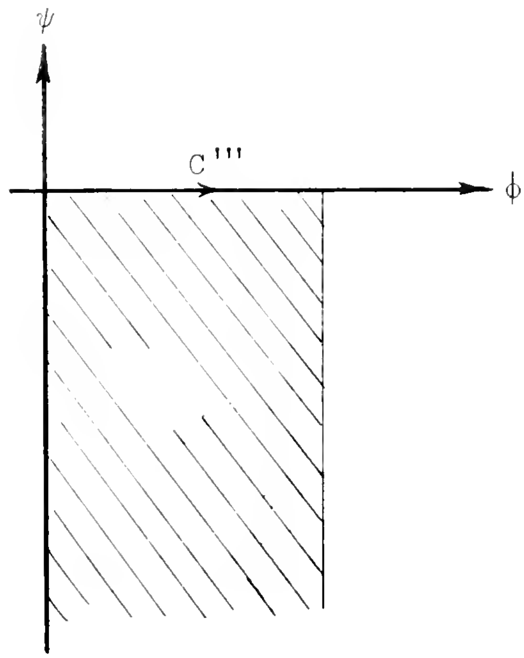


Figure 4

z-plane

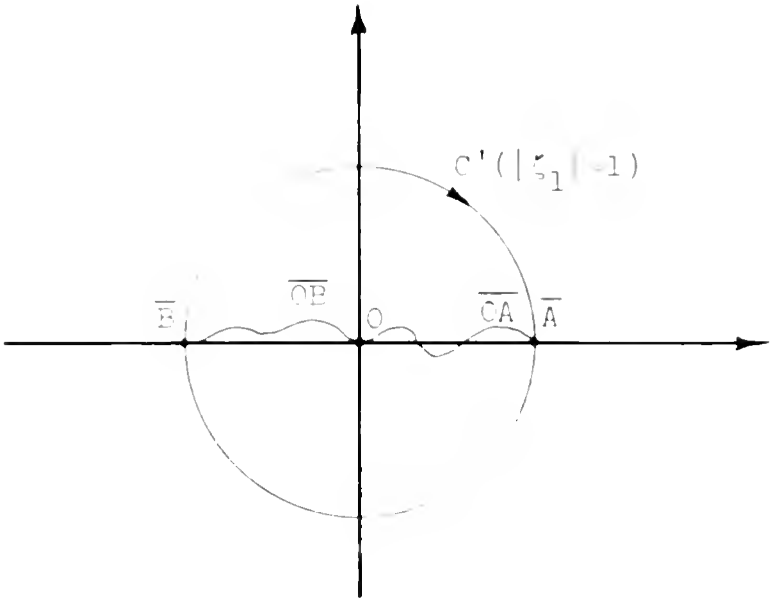


ζ-plane



Figure

ζ_1-plane



log ω-plane

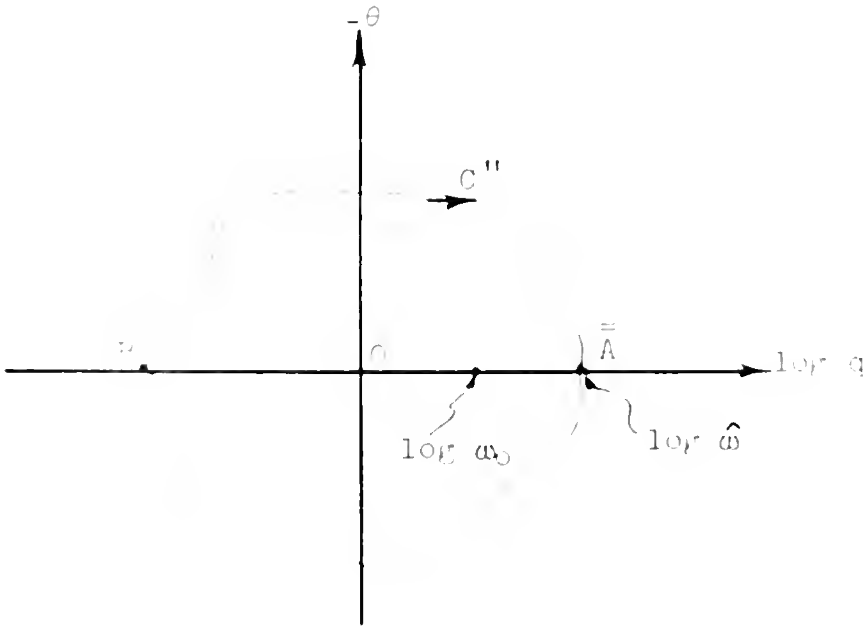


Figure 6

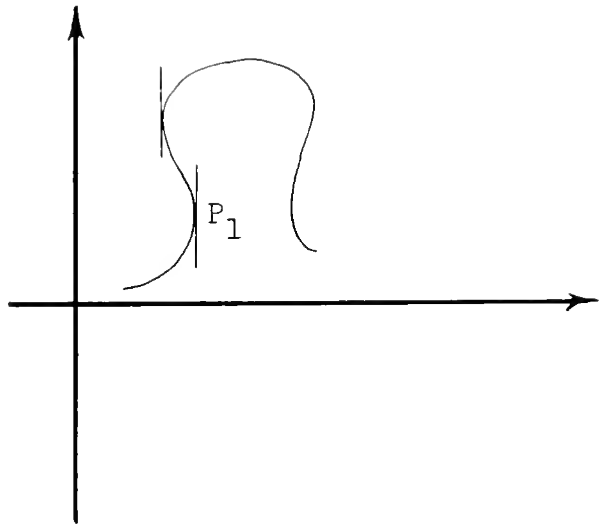
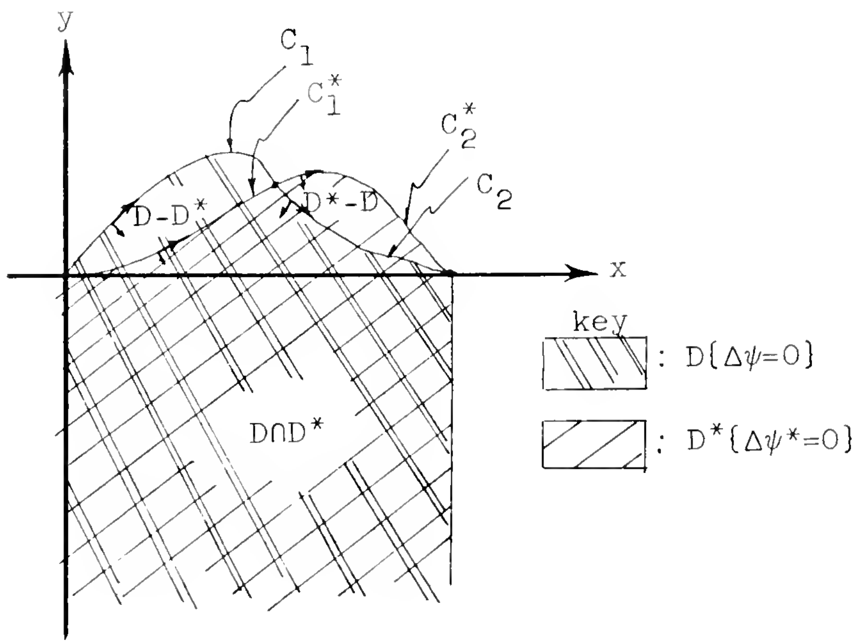


Figure 7

z-plane



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13. ABSTRACT This report deals with periodic waves on an ocean of infinite depth. The flow is assumed to be two-dimensional, incompressible, steady, and irrotational. The impossibility of the existence of an asymmetric wave is proved. This is accomplished through an application of Steiner symmetrization. Also discussed is the shape of possible periodic waves.

Using the calculus of variations, we set up an extremal problem involving the kinetic energy, an area integral, and the potential energy. For waves of small amplitude the kinetic energy is shown to be a minimum if we fix the area and the potential energy. This is accomplished by showing the first variation to be zero and the second variation to be positive. Since the kinetic energy is closely related to the Dirichlet integral, this is a generalization of the Dirichlet principle. This result is applicable in showing the existence of periodic surface waves.

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